MATHEMATICS DEPARTMENT CONTEST

Fall semester 2024

The Department of Mathematics invites all Catholic University students to compete, for the fun of it, in a mathematics contest. The contest consists of mathematical problems or puzzles which can be understood by anyone with the usual high school mathematics background. The most successful contestants will be invited to the Mathematics Department end-of-semester party to receive prizes. There will be prizes for the students who solve the most problems and for those who submit the most interesting or original solutions (even if for only one problem).

Submit your solutions by **November 19, 2024** to Dr. Alexander Levin at the Mathematics Department in Aquinas Hall, room 116. They need not be typed but should be legible and should show or explain how you solved the puzzle. You can also submit your solutions via email to levin@cua.edu by 11/19/2024. They should be arranged in one PDF file. If you take pictures, they should be very clear. Then you can convert JPG files into PDF and then merge the obtained PDF files into one PDF file (these two operations can be done online by free, just google "convert jpg to pdf" and "merge pdf files").

Problem 1. Five teams played in a soccer tournament. Each team was supposed to play one game with every other team, but some games were cancelled. It turned out that the numbers of points earned by the teams are all distinct and no team had zero points. (Teams received three points for a win, one point for a tie, and no points for a loss.) What is the minimum possible number of games that were played at the tournament? Justify your answer.

Problem 2. Prove that $a^2 + ab + b^2 \geq 3(a + b - 1)$ for any real numbers a and b.

Problem 3. It is known that all people in city A are knights who always tell the truth, while all people in city B are liars who always lie. Some people from these two cities are sitting in a plane in several rows containing four people each. A flight attendant asks each person the following question: "Is it true that the number of your city's inhabitants (including yourself) in your row is equal to the number of the other city's inhabitants in the row?". Seventy people answered "Yes" to this question. How many liars are among the plane passengers? (There are only passengers from cities A and B on the plane.)

Problem 4. Consider a right triangle ABC, where $\angle ABC = 90°$, and two points E and D on the hypotenuse AC and leg BC, respectively, such that $AE = EC$ and $\angle ADB = \angle EDC$. Find $\frac{|CD|}{|BD|}$, the ratio of the lengths of the segments CD and BD.

Problem 5. There is a set consisting of 2024 distinct real numbers such that if one replaces any number from the set by the sum of all other numbers from this set, the set will remain the same (it will still consist of the same 2024 numbers). Prove that the product of all numbers in this set is positive.

Problem 6. Is it possible that the sum of decimal digits of the square of an integer is equal to 2024? Justify your answer.