MATHEMATICS DEPARTMENT CONTEST

Spring semester 2023

The Department of Mathematics invites all Catholic University students to compete, for the fun of it, in a mathematics contest. The contest consists of mathematical problems or puzzles which can be understood by anyone with the usual high school mathematics background. The most successful contestants will be invited to the Mathematics Department end-of-semester party to receive prizes. There will be prizes for the students who solve the most problems and for those who submit the most interesting or original solutions (even if for only one problem).

Submit your solutions by **April 17, 2023** to Dr. Alexander Levin at the Mathematics Department in Aquinas Hall, room 116. They need not be typed but should be legible and should show or explain how you solved the puzzle.

Problem 1. On an island, knights always tell the truth, knaves always lie and jokers can do either. John and Steve live on the island. John said, "Steve is a knight." Steve said, "John is not a knight." Prove that at least one of these two guys is a joker who told the truth.

Problem 2. What is greater – the number of eight-digit numbers whose decimal representations contain 2, or the number of eight-digit numbers whose decimal representations do not contain 2? Justify your answer.

Problem 3. There are five points in the plane such that both coordinates of each of the points are integers. Prove that there are two points A and B among them such that both coordinates of the midpoint of the segment AB are integers.

Problem 4. Suppose that you are allowed to replace any fraction $\frac{m}{n}$ (m and n are positive integers) with either $\frac{m+n}{n}$ or $\frac{m-n}{n}$ or $\frac{n}{m}$. Can you start with $\frac{1}{2}$ and obtain $\frac{67}{91}$ using the above operations? Show the steps of such a transformation or prove that it is impossible.

Problem 5. Let x_1, x_2, x_3, x_4, and x_5 be five non-negative real numbers such that $x_1 + x_2 + x_3 + x_4 + x_5 = 1$. What is the greatest possible value of the number $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5$? Justify your answer.

Problem 6. There are $1,000,000$ tickets, each of which is assigned a sequence of six digits, from 000000 to 999999. A ticket is said to be "lucky" if the sum of its first three digits equals the sum of its last three digits. Prove that the sum of numbers of all lucky tickets is divisible by 13. (Of course, when we add, say 00*abcd* or 000*abc*, we actually add a four-digit number *abcd* or a three-digit number *abc*, etc.)