

MATHEMATICS DEPARTMENT CONTEST

Spring semester 2025

The Department of Mathematics invites all Catholic University students to compete, for the fun of it, in a mathematics contest. The contest consists of mathematical problems or puzzles which can be understood by anyone with the usual high school mathematics background. The most successful contestants will be invited to the Mathematics Department end-of-semester party to receive prizes. There will be prizes for the students who solve the most problems and for those who submit the most interesting or original solutions (even if for only one problem).

Submit your solutions by **April 10, 2025** to Dr. Alexander Levin at the Mathematics Department in Aquinas Hall, room 116. They need not be typed but should be legible and should show or explain how you solved the puzzle.

Problem 1. Let us consider 2025 integers $a_1, a_2, \dots, a_{2025}$. Let $b_1, b_2, \dots, b_{2025}$ be the same 2025 numbers written in a different order. Prove that the product $(a_1 - b_1)(a_2 - b_2) \dots (a_{2025} - b_{2025})$ is an even integer.

Problem 2. Prove that the area of a triangle whose two sides have lengths x and y cannot exceed $\frac{x^2 - xy + y^2}{2}$.

Problem 3. There are 29 students in some class. It is known that in any group of three students, there are two friends. Prove that there is a student who has more than 14 friends in the class.

Problem 4. Prove that if p and $p^2 + 2$ are prime positive integers, then the integer $p^3 + 2$ is prime as well.

Problem 5. Prove that in any set of fifty integers, one can choose several numbers whose sum is divisible by 50.

Problem 6. Consider a trapezoid ABCD with parallel sides AD and BC (the length of AD is greater than the length of BC) such that $AB = BC$, $AC = CD$, and the sum of the lengths of BC and CD is equal to the length of AD. Find the angles of the trapezoid.